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A three-dimensional law of the wall for turbulent shear flows

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An extended law of the wall is derived for three-dimensional flows. It describes the variation of the magnitude and direction of velocity close to the wall. The effects of both the pressure gradient and the inertial forces have been taken into account. The derived wall law is valid only when the deviations from the simple law of the wall are not large. The most important feature of a three-dimensional wall law is the prediction of the rotation of the velocity vector near the wall. Comparison of the flow angle variations predicted by the present wall law with the few available experimental data shows good agreement.

1. Introduction

The law of the wall describes the velocity distribution in turbulent shear flows near walls. Some information about the velocity distribution in this region can be simply obtained from dimensional analysis. If it is assumed that flow close to a smooth wall is determined completely by local conditions (wall shear stress, distance from the wall and fluid properties), a universal velocity distribution must exist near walls. In turbulent flows the viscous forces are important only in a very thin layer adjacent to the wall: the so-called viscous sublayer. In the region where the direct effect of the fluid viscosity is negligible (i.e. outside the viscous sublayer), dimensional reasoning (see e.g. Rotta 1962) leads to the conclusion that the velocity must vary logarithmically with distance from the wall. This law of the wall, which predicts a logarithmic velocity distribution outside the viscous sublayer, has been known for a long time and has been confirmed experimentally numerous times. Here it will be called the *simple law of the wall.*

An assumption in the derivation of the simple law of the wall is that the shear stress is constant and equal to the wall shear stress in the thin layer near the wall, where the law of the wall is supposed to hold. The simple law of the wall leads fundamentally to a two-dimensional velocity distribution in the plane of the wall shear stress. However, in three-dimensional boundary layers, it has been found that significant changes in flow direction may occur close to the wall (e.g. East & Pierce 1972). Changes in direction of the velocity can be predicted only by extended versions of the law of the wall, which take into account the variation of the shear stress vector with distance from the wall. This paper deals with such extensions of the simple law of the wall.

For two-dimensional flows, an extended law of the wall was deduced by

Townsend (1961) for a linearly varying shear stress $\tau = \tau_w + \text{const.} \times y$, where y is the distance from the wall. A linearly varying shear stress is found from the equation of motion, when taking into account only the effect of the pressure gradient in the flow direction and neglecting the inertial terms. When the shear stress is not constant, it is not possible to derive a velocity distribution from dimensional reasoning. A relation has to be assumed between the magnitude of the shear stress and the velocity variation. It is generally assumed that, outside the viscous sublayer, the mixing-length relation holds,

$$\tau = l^2 \rho (\partial U/\partial y)^2,$$

where l is the so-called mixing length, ρ is the fluid density and U is the fluid velocity. When $l \propto y$ and $\tau = \tau_w$, a logarithmic velocity distribution results, in accordance with the simple law of the wall. The essential assumption that will be made is that the magnitude of the mixing length remains the same when the shear stress is not constant. On the basis of this assumption, an expression for the velocity distribution can be obtained, as shown by Townsend (1961). Recently, generalizations have been given of the above procedure to three-dimensional flows, with shear stress varying linearly in magnitude and direction (Perry & Joubert 1965; Nash & Patel 1972; van den Berg 1972).

The usefulness of the extended wall laws described above appears to be limited. The reason is that only the effect of the pressure gradient on the shear stress variation with distance from the wall has been taken into account. In fact, the influence of the inertial forces on the shear stress variation appears to be far from negligible. The importance of the inertial effects is not surprising, when considering the extremely rapid increase of the velocity with distance from the wall in turbulent flows. In two-dimensional flows, the contribution of the inertial terms in the equation of motion was found to be typically about half the pressure gradient term in the region of interest (i.e. outside the viscous sublayer). To obtain more useful extended laws of the wall, it is essential, therefore, to take into account the inertial effects.

These may be estimated by substituting the velocity distribution given by the simple law of the wall into the inertial terms of the equations of motion. This is justified only when the shear stress change with distance from the wall is not large. In fact, the first-order approximation to the inertial terms for small deviations from the simple law of the wall is obtained in this way. In it, the inertial terms depend solely on the variation of wall shear stress along the surface, actually on the first derivatives of τ_w. If a higher-order approximation to the inertial terms is required, the second derivatives of the pressure, and also of τ_w, play a role, and the estimation of the inertial forces then becomes much more complicated.

As mentioned already, several attempts have been made to obtain formulae for the velocity distribution near the wall more accurate than that given by the simple law of the wall. The important inertial effects have never been taken into account in a really satisfactory manner, however, particularly for the three-dimensional case. Therefore it was decided to deduce a three-dimensional wall law along the lines just described. In view of the extensive use made of the law of the wall formula for various purposes, a three-dimensional extension of it may be supposed to have some general interest. But the immediate motive for the work was the application of the formula in a calculation method for three-dimensional turbulent boundary layers (van den Berg *et al.* 1975). It appeared essential for the construction of a satisfactory calculation method to use a good extended law of the wall. This is not very surprising, since the development of turbulent boundary layers is determined to a large extent in the law of the wall region.

In the present derivation of the wall law the first-order approximation will be taken consistently. This means that, not only the inertial effects, but also the effect of the pressure gradient will be considered only to the accuracy required for small shear-stress variations. This leads to a substantial simplification of the resulting formula, without a real loss of accuracy.

2. The mixing-length relation

As mentioned earlier, it is necessary, for the derivation of extended wall laws, to make a physical assumption about turbulent shear stress. The calculations in this paper will be based on the assumption that the mixing-length relation given below holds. In three-dimensional flows the mixing-length relation may be written, with mixing length $l = ky$ where k is the von Kármán constant, as

$$\tau_x = k^2y^2\rho\left\{\left(\frac{\partial U_x}{\partial y}\right)^2+\left(\frac{\partial U_z}{\partial y}\right)^2\right\}^{\frac{1}{2}}\frac{\partial U_x}{\partial y}, \qquad (1)$$

$$\tau_z = k^2y^2\rho\left\{\left(\frac{\partial U_x}{\partial y}\right)^2+\left(\frac{\partial U_z}{\partial y}\right)^2\right\}^{\frac{1}{2}}\frac{\partial U_z}{\partial y}. \qquad (2)$$

Here τ_x, τ_z, U_x and U_z are the components of the shear stress τ and the velocity U in x and z directions. The relation is written in such a way that the shear stress acts in the direction of the maximum rate of deformation, as in laminar flows. The relation is supposed to be valid close to the wall, but outside the viscous sublayer.

The mixing-length relation was first put forward by Prandtl nearly fifty years ago; since then it may be considered the common assumption about turbulent shear stress in the wall region. Townsend's (1961) two-dimensional extended law of the wall is based on it. According to Townsend, the mixing-length relation represents the equilibrium between production and dissipation of turbulent energy in the region close to the wall. Townsend originally included a diffusion term in the energy balance; but this received little support from later measurements. There is much experimental evidence that the mixing-length relation remains valid in a wide variety of circumstances. In the present context it is important to mention Huffman & Bradshaw (1972), who analysed a large number of measurements in turbulent flows near walls with a shear stress variation with distance from the wall. This analysis demonstrated that the mixing length remains unaltered up to substantial shear stress variations. Also, in compressible flows there is no evidence that the mixing length changes much with increasing Mach number (e.g. Maise & McDonald 1968). On the whole, the available evidence suggests that the mixing-length relation holds much better than one might expect (Patel 1973).

In the foregoing, attention was focused on the validity of the mixing-length relation in two-dimensional flows. The main additional assumption in three-dimensional flows is that the shear stress direction coincides with the direction of the maximum rate of deformation, i.e.

$$\tau_x/\tau_z = (\partial U_x/\partial y)/(\partial U_z/\partial y).$$

Such a shear stress direction follows if it is assumed that the produced turbulent shear stress (which most probably has the direction of the deformation) can be equated to the dissipated shear stress (which may be expected to occur in the shear stress direction). The assumption is in harmony with the idea of the existence of an equilibrium between turbulence production and dissipation close to the wall. Since experimental data are scarce in three-dimensional turbulent flows, no direct experimental evidence can be given to support this assumption. East (1972) suggested that the effect of diffusion of turbulence should be included close to the wall. By introducing a considerable diffusion rate (an order of magnitude larger than assumed earlier by Townsend in two-dimensional flows near walls), he obtained an essential difference between the shear stress direction and the direction of the velocity gradient, in contrast with the assumption made here. But the amount of diffusion is physically unlikely in the author's opinion. The idea was suggested to East by existing experimental evidence that substantial differences may exist between the direction of the shear stress and the velocity gradient. Such differences have indeed been found in three-dimensional turbulent boundary layers, but not in the thin region close to the wall, where the law of the wall is supposed to hold. An accurate determination of shear stress in this region will be extremely difficult. Although not much certainty exists about the shear stress direction, it seems most sensible to assume for the present that the shear stress acts in the direction of the velocity gradient close to the wall.

3. The derivation of the law of the wall

The equations of motion for thin shear layers read

$$\rho U_x \frac{\partial U_x}{\partial x} + \rho U_y \frac{\partial U_x}{\partial y} + \rho U_z \frac{\partial U_x}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_x}{\partial y}, \quad (3)$$

$$\rho U_x \frac{\partial U_z}{\partial x} + \rho U_y \frac{\partial U_z}{\partial y} + \rho U_z \frac{\partial U_z}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_z}{\partial y}. \quad (4)$$

The left-hand sides of the equations are $O(y^2)$ close to a wall, if y is the distance from the wall. For small y, therefore, the shear stress gradient and pressure gradient may be equated, so that the shear stress distribution close to a wall may be written as

$$\tau_x = \tau_{w_x} + \frac{\partial p}{\partial x} y, \quad \tau_z = \tau_{w_z} + \frac{\partial p}{\partial z} y. \quad (5), (6)$$

The validity of the above equations is restricted to an extremely thin layer adjacent to the wall, usually even thinner than the viscous sublayer. To extend the validity to regions of more practical interest, it is necessary to include the left-hand sides of equations (3) and (4) (i.e. to estimate the contribution of the inertial terms). As a first approximation, it may be assumed, for the purpose of such an estimate, that the velocity is determined completely by the wall distance and the local wall shear stress vector. Consequently, the contribution of the inertial terms can be deduced from the local wall shear stress gradients, which are supposed to be known. The foregoing means that the simple law of the wall will be used here to correct the simple law of the wall for the shear stress variations due to the inertial effects. This may be done only when the shear stress variation with distance from the wall is small, i.e. when

$$|\tau_x - \tau_{w_x}| \ll \tau_w \quad \text{and} \quad |\tau_z - \tau_{w_z}| \ll \tau_w.$$

It is useful to define a skin-friction velocity vector in the direction of the wall shear stress with a magnitude $u_\tau = (\tau_w/\rho)^{\frac{1}{2}}$ and components u_{τ_x} and u_{τ_z}. Writing $y^+ = yu_\tau/\nu$, where ν is the kinematic viscosity, the simple law of the wall may be expressed as

$$U_x = u_{\tau_x} f(y^+), \quad U_z = u_{\tau_z} f(y^+). \tag{7), (8}$$

The velocity U_y normal to the wall can be found by integration of the continuity equation:

$$U_y = -\int_0^y \left(\frac{\partial U_x}{\partial x} + \frac{\partial U_z}{\partial z}\right) dy. \tag{9}$$

The contribution of the inertial terms is now obtained by substitution of (7)–(9) in the left-hand sides of (3) and (4). With some algebra, a complicated expression is derived for the inertial terms. This expression becomes much simpler when one of the co-ordinate axes coincides with the direction of the wall shear stress. Henceforth the x axis will be taken in the wall shear stress direction at the position considered. The shear stress distribution, with the inertial effects taken into account, then becomes

$$\frac{\tau_x}{\tau_w} = 1 + \frac{\nu}{\rho u_\tau^3} \frac{\partial p}{\partial x} y^+ + \frac{\nu}{u_\tau^2} \frac{\partial u_{\tau_x}}{\partial x} \int_0^{y^+} f^2\, dy^+ + \frac{\nu}{u_\tau^2} \frac{\partial u_{\tau_z}}{\partial z} \left[f \int_0^{y^+} f\, dy^+ - \int_0^{y^+} f^2\, dy^+ \right], \tag{10}$$

$$\frac{\tau_z}{\tau_w} = \frac{\nu}{\rho u_\tau^3} \frac{\partial p}{\partial z} y^+ + \frac{\nu}{u_\tau^2} \frac{\partial u_{\tau_z}}{\partial x} \int_0^{y^+} f^2\, dy^+. \tag{11}$$

When investigating the numerical magnitude of the integrals in (10) and (11), it appears that the integral expression of the last term in (10) is very small for all practical values of y^+ (see van den Berg 1972). It is justifiable, therefore, to omit this term, for reasons of simplicity. To evaluate the integrals, the well-known log law $f(y^+) = k^{-1}(\ln y^+ + A)$ may be substituted. This gives (taking $A = 2$, which is very near the usually assumed value)

$$\int_0^{y^+} f^2\, dy^+ = k^{-2} (\ln y^+ + 1)^2 y^+ + \frac{y^+}{k^2} + c.$$

The constant c has been added to account for the error made by using the log law also in the viscous sublayer. The above formula was compared with a numerical integration of f^2, using the tabulated function of Coles (1955). The tabulated function f describes the correct velocity distribution in the viscous sublayer with a smooth transition to the log law. It appears (van den Berg 1972) that the contribution of c, and also of y^+/k^2, may be neglected for $y^+ > 30$ (i.e. outside the viscous sublayer). Consequently, the integral can be approximated by

$$\int_0^{y^+} f^2\,dy^+ = k^{-2}(\ln y^+ + 1)^2 y^+. \tag{12}$$

For convenience, the parameters

$$\alpha_x = \frac{\nu}{\rho u_\tau^3}\frac{\partial p}{\partial x} \quad \text{and} \quad \alpha_z = \frac{\nu}{\rho u_\tau^3}\frac{\partial p}{\partial z}, \tag{13}$$

$$\beta_x = \frac{\nu}{u_\tau^2}\frac{\partial u_{\tau x}}{\partial x} = \frac{\nu}{u_\tau^2}\frac{\partial u_\tau}{\partial x} \quad \text{and} \quad \beta_z = \frac{\nu}{u_\tau^2}\frac{\partial u_{\tau z}}{\partial x} = \frac{\nu}{u_\tau}\frac{\partial \phi_\tau}{\partial x} \tag{14}$$

are introduced, where ϕ_τ is the wall shear stress angle. It may be helpful to recall that all quantities are defined in a Cartesian co-ordinate system with the x axis in the direction of the wall shear stress. Equations (10) and (11) can now be written as

$$\frac{\tau_x}{\tau_w} = 1 + \alpha_x y^+ + \beta_x \frac{(\ln y^+ + 1)^2 y^+}{k^2}, \tag{15}$$

$$\frac{\tau_z}{\tau_w} = \alpha_z y^+ + \beta_z \frac{(\ln y^+ + 1)^2 y^+}{k^2}. \tag{16}$$

From the found shear stress distribution, the velocity distribution has to be deduced. For that purpose, the mixing-length relation, given by (1) and (2), will be applied. When one takes

$$\tau_x - \tau_w \ll \tau_w \quad \text{and} \quad \tau_z \ll \tau_w,$$

an assumption made earlier, (1) and (2) may be rewritten:

$$\frac{\partial U_x}{\partial y} = \frac{1}{ky}\frac{\tau_x}{\rho^{\frac{1}{2}}|\tau|^{\frac{1}{2}}} \approx \frac{1}{ky}\left(\frac{\tau_x}{\rho}\right)^{\frac{1}{2}}, \tag{17}$$

$$\frac{\partial U_z}{\partial y} = \frac{1}{ky}\frac{\tau_z}{\rho^{\frac{1}{2}}|\tau|^{\frac{1}{2}}} \approx \frac{1}{ky}\frac{\tau_z}{(\rho\tau_w)^{\frac{1}{2}}}. \tag{18}$$

Substitution of the shear stress distribution given by (15) and (16) leads to a three-dimensional extended law of the wall. Integration, after expanding in power series for small $\alpha_x y^+$, etc., and retaining the leading terms, results in the simple formulae

$$U_x^+ = \frac{1}{k}\left[\ln y^+ + A + \tfrac{1}{2}\alpha_x y^+ + \tfrac{1}{2}\beta_x \frac{(\ln y^+)^2 y^+}{k^2}\right], \tag{19}$$

$$U_z^+ = \frac{1}{k}\left[\alpha_z(y^+ + b) + \beta_z \frac{(\ln y^+)^2 y^+}{k^2}\right], \tag{20}$$

where $(\ln y^+)^2 + 1$ is approximated by $(\ln y^+)^2$. Here $U_x^+ = U_x/u_\tau$ is the dimensionless velocity in the wall shear stress direction, and $U_z^+ = U_z/u_\tau$ that in the crosswise direction. The constants A and b are integration constants, which appear because the mixing-length relation does not hold up to the wall. Very close to the wall, the turbulent shear stresses are suppressed, and the viscous shear stresses become important. The velocity increment in the viscous sublayer and the change in direction are accounted for by the constants A and b. Some remarks on these constants will be made shortly.

Instead of the velocity components U_x and U_z, the magnitude and direction of the velocity may be given. Within the approximation considered here (small $\alpha_x y^+$, etc.), the absolute value of the velocity is equal to U_x. The flow angle ϕ, relative to the wall shear stress angle, may be written as

$$\phi = \frac{\alpha_z(y^+ + b) + \beta_z(\ln y^+)^2 y^+/k^2}{\ln y^+ + A}. \tag{21}$$

For $\alpha_x = \beta_x = 0$, (19) reduces to the well-known log law. Equation (19) with $\beta_x = 0$ may be compared with Townsend's two-dimensional extended law of the wall, which neglects inertial effects:

$$U_x^+ = \frac{1}{k}[\ln y^+ + A - 2\ln \tfrac{1}{2}\{(1+\alpha_x y^+)^{\frac{1}{2}} + 1\} + 2(1+\alpha_x y^+)^{\frac{1}{2}} - 2]. \tag{22}$$

It is easy to show that (22) reduces to (19) with $\beta_x = 0$ for $\alpha_x y^+ \ll 1$, which has to be the case. In figure 1, the velocity distributions according to both formulae are compared for a typical value of α_x. The logarithmic velocity distribution of the simple law of the wall has also been plotted. It is evident that the agreement between (19) and (22) remains good up to surprisingly large values of $\alpha_x y^+$. Even for $\alpha_x y^+ = 1$, which means that the shear stress is doubled in the region considered, the discrepancy is still negligible, while the difference between the velocities given by the extended wall laws and the simple wall law is substantial. The error due to the assumption that shear stress variation with distance from the wall is small is rather less than one might have expected.

Townsend assumed the value of A to be independent of the pressure gradient parameter α_x, so that the constant A in his formula may be equated to the constant A in the simple law of the wall, which is known from experiment. An attempt will be made here to evaluate the effect of a pressure gradient on A, by assuming purely laminar flow near the wall, with an abrupt transition to turbulent flow at a certain value of $(y/\nu)(\tau/\rho)^{\frac{1}{2}}$. In laminar flow with a pressure gradient, the velocity in the wall shear stress direction and perpendicular to it may be written as

$$U_x^+ = y^+(1 + \tfrac{1}{2}\alpha_x y^+), \quad U_z^+ = \tfrac{1}{2}\alpha_z(y^+)^2. \tag{23), (24}$$

For $\alpha_x = 0$, agreement must exist with the simple law of the wall, in which case $A \approx 2$. It appears that, to obtain $A = 2$, one should pass from the laminar to the turbulent equations at $(y/\nu)(\tau/\rho)^{\frac{1}{2}} \approx 11$. If (23) is fitted to (19) at $(y/\nu)(\tau/\rho)^{\frac{1}{2}} = 11$, it is found that A is independent of α_x, when $O(\alpha^2)$ or higher-order terms are neglected. So, as a first approximation, A may be taken equal to the value in the simple law of the wall as in Townsend's analysis. The same result was obtained by Mellor (1966), who established the effect of a pressure gradient on A in a more detailed fashion, assuming a gradual instead of abrupt transition from the fluid

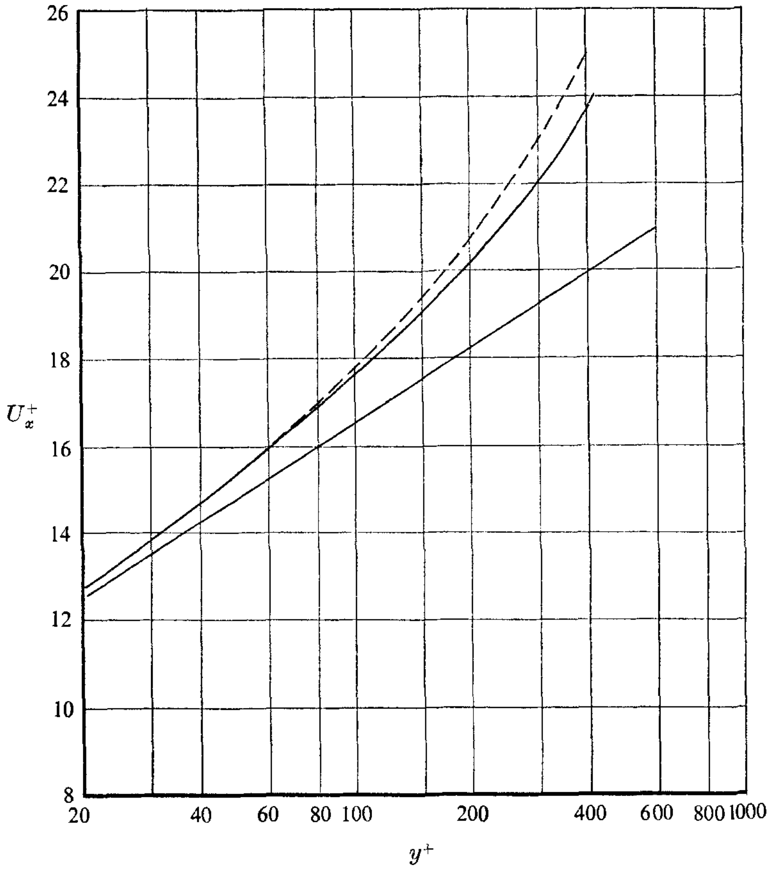

FIGURE 1. Comparison of velocity variations, close to the wall in two-dimensional flow, predicted by various versions of the law of the wall. $\alpha_x = 10^{-2}$. Extended laws of the wall (upper curves): ---, (19) with $\beta_x = 0$, viz. $U_x^+ = k^{-1}(\ln y + A + \frac{1}{2}\alpha_x y^+)$; ——, Townsend (see (22)). Simple law of the wall (lower curve): $U_x^+ = k^{-1}(\ln y^+ + A)$.

viscosity to an eddy viscosity function that corresponded to the mixing-length concept outside the viscous sublayer. If (24) is fitted to (20) at $(y/\nu)(\tau/\rho)^{\frac{1}{2}} = 11$, one obtains $b \approx 13$, neglecting again $O(\alpha^2)$ or higher-order terms. Further details about the determination of the constants A and b are given in van den Berg (1972). The influence of the inertial forces (i.e. of β_x and β_z) on the constants has been neglected, because inertial effects are most important further away from the wall, and should therefore not affect these constants very much.

4. The effect of compressibility

The influence of the compressibility of the fluid will be taken into account here only to the first-order approximation for small density variations. In that case, the compressibility effect may be superposed on the effect of the shear stress variations with distance from the wall, which was established to the first-order approximation in § 3. So it suffices to consider the influence of compressibility in a constant shear stress layer. Furthermore, the discussion will be limited to flows with zero heat transfer to or from the wall.

Near adiabatic walls the distribution of the absolute temperature T satisfies quite well the Crocco relation

$$\frac{T}{T_w} = 1 - \frac{r}{2C_p}\frac{U^2}{T_w}, \tag{25}$$

where r is the recovery factor ($r \approx 0{\cdot}9$), and C_p is the specific heat at constant pressure. The density variation across the shear layer follows from $\rho_w T_w = \rho T$. An appropriate compressibility parameter near walls is the skin-friction Mach number $M_\tau = u_\tau/a_w$, where $u_\tau = (\tau_w/\rho_w)^{\frac{1}{2}}$ and a_w is the speed of sound at the wall. Equation (25) can now be written as

$$\rho_w/\rho = 1 - \tfrac{1}{2}r(\gamma-1)\,M_\tau^2(U^+)^2, \tag{26}$$

where γ is the ratio of specific heats. It will again be assumed that the mixing-length relation holds. When $\tau = \tau_w$, the relation becomes, in compressible flows,

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{ky}\left(\frac{\rho_w}{\rho}\right)^{\frac{1}{2}}. \tag{27}$$

Substitution of (26) into (27) and integration gives, with the assumption that density variation is small,

$$U^+ = \frac{1}{k}\left[\ln y^+ + A - \frac{r(\gamma-1)}{12}M_\tau^2\frac{(\ln y^+ + 2)^3}{k^2}\right]. \tag{28}$$

This equation is in agreement with the compressible wall laws for the case of adiabatic walls, given by Van Driest (1951) and Rotta (1960), at least for $M_\tau \ll 1$. It appears that agreement remains good up to $M_\tau \approx 0{\cdot}15$. Much larger values of M_τ will seldom occur in non-hypersonic shear layers. Moreover, if M_τ is not small, the temperature variations in the wall law region will be so large that it becomes very questionable whether the mixing-length relation is still valid. At $M_\tau = 0{\cdot}05$ and $y^+ = 100$, (28) gives a 2 % decrease in velocity, as compared with the velocity in incompressible flow. So it appears that the influence of compressibility on the velocity distribution is rather small; and, at moderate Mach numbers (say $M_\tau < 0{\cdot}05$, which holds in practically all subsonic flows), the incompressible law of the wall, substituting the fluid properties at the wall, will give very reasonable results.

To apply a correction for compressibility effects to the general three-dimensional law of the wall given in § 3, the last term of (28) has to be added to (19), which describes the variation of the velocity in wall shear stress direction. No compressibility effect on the direction of the velocity occurs within the accuracy considered here.

5. Comparison with experimental results

A comparison of the velocity distribution predicted by a law of the wall with measurements is hampered by the fact that the law of the wall is valid only in a very thin region adjacent to the wall. In shear layers of laboratory scale, the

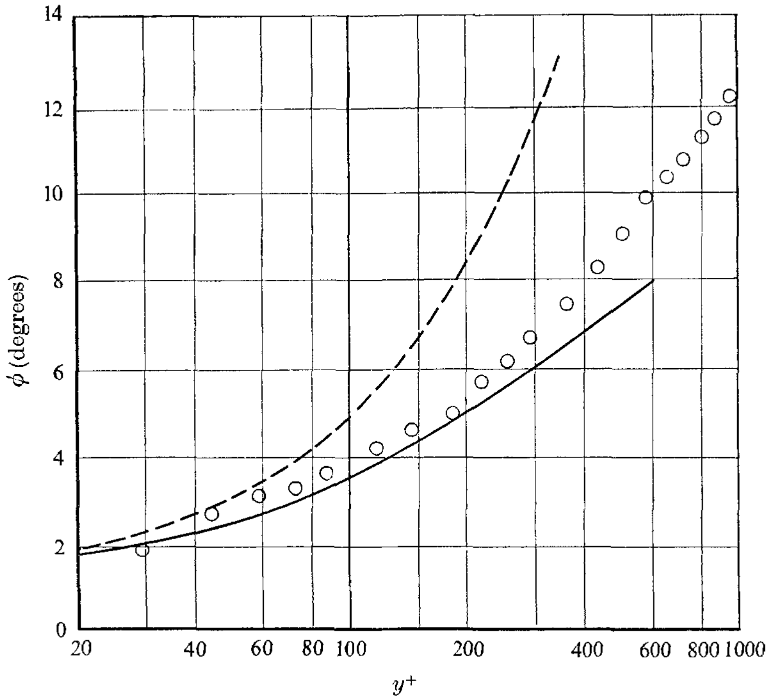

FIGURE 2. Comparison of measured with predicted variation of flow angle ϕ close to the wall. Station 5 of van den Berg's & Elsenaar's (1972) experiment; $\alpha_z = 5{\cdot}04 \times 10^{-3}$; $\beta_z = -13{\cdot}1 \times 10^{-6}$. ○, measurement. Prediction: ---, no inertial terms; ——, with inertial terms, (21).

use of a measuring device of very small dimensions, such as a hot wire, is required to determine the velocity with acceptable accuracy. Another problem is the accurate measurement of the magnitude and direction of the wall shear stress, which are essential data. Moreover, interest is focused here on the relatively small deviations from the simple law of the wall that are predicted by extended wall laws. Consequently, comparisons with experiment are often inconclusive. The most significant deviation from the simple law of the wall is the predicted rotation of the velocity vector by the three-dimensional law of the wall derived in § 3. A flow angle variation of the order of 5° may well occur in the law of the wall region, and this should be measurable. It seems sensible, therefore, to restrict comparisons with experiment to the variation of the velocity direction close to the wall.

Van den Berg & Elsenaar (1972) have carried out velocity measurements with hot wires up to very close to the surface in an incompressible three-dimensional turbulent boundary layer. (See also van den Berg *et al.* 1975.) The wall shear stress direction was determined by them with Stanton-type wall Pitots. This was done by rotating the wall Pitots, then establishing the symmetry line of the data. In fact, of course, a sort of mean flow direction over the wall Pitot height is found in this way. Since the flow angle variation is large close to the wall, a substantial systematic error may thus be made. Therefore, recently an attempt has been made to determine in the same test set-up the wall shear stress angle from oil flow

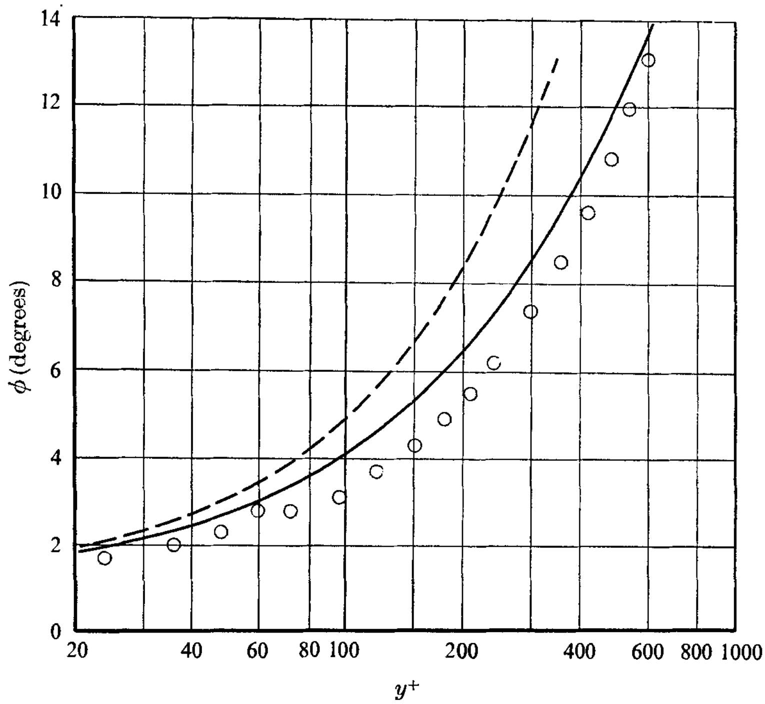

FIGURE 3. Comparison of measured with predicted variation of flow angle ϕ close to the wall. Station 7 of van den Berg's & Elsenaar's (1972) experiment; $\alpha_z = 4{\cdot}95 \times 10^{-3}$; $\beta_z = -7{\cdot}0 \times 10^{-6}$. Symbol key as for figure 2.

patterns. One may expect these to be created by the wall shear stresses themselves; so they should give a better indication of their direction. At low speeds, as in the present experiment, oil flow patterns are often too irregular to determine wall shear stress angles with reasonable accuracy. But a special oil mixture was used here, with very fine grains, which resulted in tiny but clear traces of the wall streamlines. (This mixture was kindly made available by the Department of Aeronautical Engineering, Delft University of Technology.) The wall shear stress angles, obtained from the oil flow patterns, appeared to be reproducible within $\pm 0{\cdot}5°$ (see van den Berg 1972). At the measuring stations considered, the angles appear to exceed the wall shear stress angles, found earlier with the wall Pitots, by about 2°. This difference is actually about the estimated error of the wall Pitot measurements. It is believed that the new shear stress angles are the most reliable, and the comparisons that will be made with the theory are based on them.

In figures 2 and 3, the measured flow angle variation with distance from the wall is compared with the variation according to (21). The two measuring stations, at which the comparisons are made, were chosen because they are situated in the region with the largest flow angle variations near the wall in this test set-up. In both cases, agreement between theory and experiment is seen to be very reasonable, considering the accuracy of the establishment of the wall shear stress direction. At high values of y^+, deviations may become apparent, since (21) does not hold for large changes in shear stress, which occur when $\alpha_z y^+$

becomes large. In both figures, a curve is included which gives the flow angle variation that would have been predicted neglecting the inertial effects (i.e. if $\beta_z = 0$ in (21)). It is evident that the contribution of the inertial terms is far from negligible.

6. Conclusions

It is possible to derive a three-dimensional law of the wall, valid to the first-order approximation for small deviations from the simple two-dimensional law of the wall. This law of the wall takes full account of inertial effects, which are not negligible at all compared with the effect of a pressure gradient in the region of interest. The most significant feature of this three-dimensional law of the wall is the prediction of the rotation of the velocity vector close to the wall. Comparisons of the predicted flow angle variation with the few experimental data available show good agreement.

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